

Die Wirkung phänomenologischer Dreikörperkräfte

auf die Absättigung der Bindungsenergien

leichter Kerne

Inaugural - Dissertation

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# **The Effect of Phenomenological Three-body Forces on the Saturation of Binding Energies of Light Nuclei**

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With 5 Figures in the text

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Mit Hilfe eines empirischen, zweiparametrischen Ansatzes vom Gauß-Typ für ein Dreikörperpotential wird der Einfluß solcher Kräfte, zusätzlich zu den nicht-sättigenden, rein anziehenden Zweikörperkräften, auf Bindungsenergie und stabilen Radius leichter Kerne mit abgeschlossenen Schalen, bzw. abgeschlossenen  $\pm 1$ -Konfigurationen untersucht. Die Rechnung basiert auf der Verwendung von Schalenmodellwellenfunktionen ohne Konfigurationsmischung. Es existiert keine Wahl der Parameter, die Bindungsenergie und stabilen Kernradius simultan in Übereinstimmung mit dem Experiment bringt. Da das benutzte Näherungsverfahren bei den behandelten Kernen für reguläre Potentiale ausreichend sein sollte, ist das Ergebnis ein Argument gegen die Vermutung, nicht-singuläre, abstoßende Dreikörperkräfte seien für die Absättigung der Kernkräfte verantwortlich. Hält man den Kernradius, wie bei den üblichen Zweikörperkraft-Rechnungen, konstant, so ergibt sich bei der Anpassung der Bindungsenergie an den experimentellen Wert für die Stärke  $V_0'$  der Dreikörperkraft bei einer angenommenen Reichweite  $\approx 10^{-13}$  cm der Wert  $V_0' \approx 140$  MeV.

## I. Introduction

It is well known that the observed saturation of nuclear density and nuclear binding energies, i.e. the close proportionality of the nuclear volume and the binding energy with mass number  $A$  cannot be explained by ordinary (short range and attractive) nuclear forces alone<sup>1</sup>. They lead to binding energies proportional to  $A^2$  and much too small nuclear radii (collapsed state). A mixture of ordinary and exchange forces could solve the difficulty, but the amount of exchange forces warranted by study of nucleon scattering at high energies (Serber mixture<sup>2</sup> — equal proportion of ordinary and exchange forces) is insufficient to explain saturation.

<sup>1</sup> BLATT, J.M., and V. WEISSKOPF: Theoretical Nuclear Physics. New York: John Wiley 1952.

<sup>2</sup> SERBER, R.: Phys. Rev. **72**, 114 (1947).

In recent years the effect of a repulsive hard core of range  $r_0 \approx 0.5$  to  $0.6 \times 10^{-13}$  cm, first suggested by JASTROW<sup>3</sup> on empirical grounds, has been studied in detail<sup>4</sup>. The idea of a hard core also follows from pseudoscalar meson theory<sup>5</sup> and it appears that with its help the saturation of "nuclear matter" can be satisfactorily explained<sup>6,7</sup>. However, the calculations with a hard core for a finite nucleus is beset with a number of difficulties and no final conclusions can be drawn as yet.

An alternative approach for the solution of the problem of nuclear saturation has been the introduction of many-body forces, i.e. the assumption that the interaction between a pair of nucleons is dependent on the presence of surrounding nucleons. Various authors<sup>8</sup> have investigated the effect of n-body forces from a field theoretic<sup>9,10,11,12</sup> approach, using for example a non-linear meson theory<sup>13,14</sup>. In a recent investigation DRELL and HUANG<sup>15</sup> have calculated the effect of three- and more-body forces using a pseudoscalar meson theory with pseudoscalar coupling and assuming the Lévy potential<sup>5</sup> with a repulsive hard core, i.e. assuming the interaction potential to be infinite for  $r < r_e$ , where  $r_e$  is the hard core radius. According to their theory, n-body forces are repulsive or attractive according as n is odd or even, respectively. For n greater than 4 or 5 the forces are negligible. The three-body potential of DRELL and HUANG is repulsive, spin and charge independent and a function of the interparticle distances only; its range is one third of the range of two body forces.

In the following we shall investigate the effect of an empirical repulsive, Gauss type three-body force — the simplest type of a many-body force — on nuclear saturation, i.e. examine how far the binding energies and nuclear radii obtained are in agreement with experiment. We shall assume the three-body potential to exist in addition to the usual two-body forces including exchange forces and spin-orbit interaction, but without repulsive hard core. We shall neglect the tensor force whose

<sup>3</sup> JASTROW, R.: Phys. Rev. **79**, 389; **81**, 165 (1950/51).

<sup>4</sup> BETHE, H.: Phys. Rev. **103**, 115-3 (1956) and further references quoted there.

<sup>5</sup> LÉVY, M.: Phys. Rev. **88**, 725 (1952).

<sup>6</sup> BRUECKNER, K.A., and J.L. GAMMEL: Phys. Rev. **109**, 1023 (1958).

<sup>7</sup> GOMES, L.C., J.D. WALECKA and V.F. WEISSKOPF: To be published in Annals of Physics.

<sup>8</sup> JANOSSY, L.: Proc. Cambridge Phil. Soc. **35**, 616 (1939).

<sup>9</sup> PRIMAKOFF, H., and T. HOLSTEIN: Phys. Rev. **55**, 1218 (1939).

<sup>10</sup> THIRRING, W.E.: Z. Naturforsch. **7a**, 63 (1952).

<sup>11</sup> MITTELSTAEDT, P.: Z. Physik **137**, 545 (1957).

<sup>12</sup> JOHNSON, M.H., and E. TELLER: Phys. Rev. **98**, 783 (1955).

<sup>13</sup> SCHIFF, L.I.: Phys. Rev. **84**, 1, 10 (1951); **86**, 856 (1952).

<sup>14</sup> MALENKA, B.I.: Phys. Rev. **86**, 68 (1952).

<sup>15</sup> DRELL, S.D., and K. HUANG: Phys. Rev. **91**, 1527 (1953).

contribution to the binding energy should be small. We shall choose some light nuclei with closed shell or closed shell  $\pm 1$  configurations, so that their wave function may be approximately represented by a single Slater determinant and try to adjust the parameters of the three-body potential to give correct binding energies and nuclear radii.

The neglect of many-body forces of higher order ( $n=4, 5, \dots$ ) in a phenomenological treatment can be justified from the following consideration: Various predictions of the shell model, such as spins and magnetic moments, are in good agreement with experiment and indicate the presence of definite orbitals inside the nucleus. It is hard to understand how these orbitals could persist, if many-body forces of high order contributed appreciably. One would then expect that nuclear matter rather forms some sort of "nuclear jam" like that assumed by JOHNSON and TELLER<sup>8</sup>. In such a situation there could occur "excitations" which behave in many respects like single particles. But they would probably not have the properties of free nucleons, a fact which is very likely to follow from a comparison between the shell model predictions and e.g. the empirical magnetic moments of nuclei.

We shall see in the following that a complete solution of the problem, i.e. correct B. E. at stable radius and stable radius equals the empirical radius, cannot be obtained within the framework of our assumptions and approximations. The most important of these are the neglect of the hard core and the use of uncorrelated independent particle wave functions.

## II. Method of Calculation

**The wave functions.** The binding energy of a nucleus is the expectation value of the assumed nuclear Hamiltonian with respect to wave functions which are chosen to approximate the eigenfunction of this Hamiltonian. We shall assume for the wave function  $\Psi$  a Slater determinant built up from normalised single particle wave functions. The latter will be taken as eigenfunctions of an oscillator potential

$$V(r) = \frac{1}{2} \frac{\alpha^2 \hbar^2 r^2}{M}, \quad \alpha = \frac{M\omega}{\hbar} \quad (1)$$

( $M$  = nucleon mass,  $\omega$  = circular frequency of the oscillator)

and an additional spin orbit coupling term of the form  $\frac{\lambda}{(Mc)^2} = \frac{1}{r} \cdot \frac{dV}{dr} (\vec{l} \vec{s})$  where  $\lambda$  is a numerical constant ( $\lambda \simeq 20$ ).  $l$  and  $j$  are both good quantum numbers in that case. We give below the single particle wavefunctions for the first three shells explicitly. They have been constructed in such a manner that they are simultaneous eigenfunctions of both  $\vec{j}^2$  and  $\vec{j}_z$ .

## 1st Shell

$$1 S_{11} = f(r) \delta^+$$

$$2 S_{1-1} = f(r) \delta^-$$

## 2nd Shell

$$2 P_{11} = \sqrt{\frac{2}{3}} \cdot \sqrt{\alpha} \cdot f(r) [(x + i y) \delta^- + z \delta^+]$$

$$2 P_{1-1} = \sqrt{\frac{2}{3}} \cdot \sqrt{\alpha} \cdot f(r) [(x - i y) \delta^+ - z \delta^-]$$

$$2 P_{33} = \sqrt{\alpha} \cdot f(r) (x + i y) \delta^+$$

$$2 P_{3-3} = \sqrt{\alpha} \cdot f(r) (x - i y) \delta^-$$

$$2 P_{31} = \sqrt{\frac{1}{3}} \cdot \sqrt{\alpha} \cdot f(r) [(x + i y) \delta^- - 2z \delta^+]$$

$$2 P_{3-1} = \sqrt{\frac{1}{3}} \cdot \sqrt{\alpha} \cdot f(r) [(x - i y) \delta^+ + 2z \delta^-]$$

## 3rd Shell

$$3 S_{11} = \sqrt{\frac{3}{2}} \cdot \alpha \cdot f(r) \left[ \frac{1}{\alpha} - \frac{2}{3} (x^2 + y^2 + z^2) \right] \delta^+$$

$$3 S_{1-1} = \sqrt{\frac{3}{2}} \cdot \alpha \cdot f(r) \left[ \frac{1}{\alpha} - \frac{2}{3} (x^2 + y^2 + z^2) \right] \delta^-$$

$$3 D_{31} = \sqrt{\frac{2}{15}} \cdot \alpha \cdot f(r) [(2z^2 - x^2 - y^2) \delta^+ + 3z(x + i y) \delta^-]$$

$$3 D_{3-1} = \sqrt{\frac{2}{15}} \cdot \alpha \cdot f(r) [(2z^2 - x^2 - y^2) \delta^- - 3z(x - i y) \delta^+]$$

$$3 D_{33} = \sqrt{\frac{2}{5}} \cdot \alpha \cdot f(r) [(x + i y)^2 \delta^- + z(x + i y) \delta^+]$$

$$3 D_{3-3} = \sqrt{\frac{2}{5}} \cdot \alpha \cdot f(r) [(x - i y)^2 \delta^+ - z(x - i y) \delta^-]$$

$$3 D_{51} = \sqrt{\frac{1}{5}} \cdot \alpha \cdot f(r) [(2z^2 - x^2 - y^2) \delta^+ - 2z(x + i y) \delta^-]$$

$$3 D_{5-1} = \sqrt{\frac{1}{5}} \cdot \alpha \cdot f(r) [(2z^2 - x^2 - y^2) \delta^- + 2z(x - i y) \delta^+]$$

$$3 D_{53} = \sqrt{\frac{1}{10}} \cdot \alpha \cdot f(r) [(x + i y)^2 \delta^- - 4z(x + i y) \delta^+]$$

$$3 D_{5-3} = \sqrt{\frac{1}{10}} \cdot \alpha \cdot f(r) [(x - i y)^2 \delta^+ + 4z(x - i y) \delta^-]$$

$$3 D_{55} = \sqrt{\frac{1}{2}} \cdot \alpha \cdot f(r) (x + i y)^2 \delta^+$$

$$3 D_{5-5} = \sqrt{\frac{1}{2}} \cdot \alpha \cdot f(r) (x - i y)^2 \delta^-$$

(2)

where  $\delta^+$  and  $\delta^-$  are the spin functions and

$$f(r) = \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\alpha}{2} r^2\right).$$

The spectroscopic notation  $2P_{31}$ , for example, signifies an eigenfunction of the 2nd Shell with  $l=1$ ,  $j=\frac{3}{2}$  and  $j_z=\frac{1}{2}$ .

Using this set of wave functions and the usual general two-body interaction consistent with the scattering data (nearly Serber mixture of ordinary and exchange forces) v. SIEMENS<sup>16</sup> has calculated the binding energies  $E_B^{2\text{ body}}(\alpha)$  as a function of the radius parameter  $\alpha$  (Nuclear radius  $\propto \frac{1}{\sqrt{\alpha}}$ ) for light nuclei upto  $A=40$ . He obtains in general

$$|E_B^{2\text{ body}}(\alpha_0)| > |E_B^{(\text{expt})}|$$

where  $\alpha_0$  is the experimental value of  $\alpha$ , obtained from the difference in binding energies of mirror nuclei, attributing this effect to a difference in the Coulomb energies. For  $O^{16}$  the discrepancy is about 20%. The stable  $\alpha$  (obtained from the maximum of the  $E_B^{2\text{ body}}-\alpha$  graph) comes out much too high which indicates the aforementioned nuclear collapse.

The repulsive three-body force to be introduced should be such that the total binding energy calculated from 2-body plus 3-body forces should satisfy the following two conditions:

$$\left. \begin{aligned} \text{(I)} \quad & \frac{\partial E_B^{\text{total}}}{\partial \alpha} = 0 \quad \text{for experimental value of } \alpha, (\alpha_0 \text{ say}) \\ \text{i.e. the theoretical stable radius should agree with the experi-} \\ & \text{mental value of the radius} \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} \text{(II)} \quad & E_B^{\text{total}}(\alpha_0) = \text{experimental value of the binding energy} \\ \text{i.e. the nucleus should have the correct binding energy for the} \\ & \text{stable radius [which according to (I) equals the empirical radius].} \end{aligned} \right\} \quad (4)$$

**The three-body potential.** We assume the following purely Wigner type potential acting between all possible nuclear trios in the nucleus:

$$\left. \begin{aligned} V^{3\text{ body}} &= \sum_{i < j < k}^A V(i, j, k) \\ V(1, 2, 3) &= V_0' e^{-\beta' [(r_1-r_2)^2 + (r_2-r_3)^2 + (r_1-r_3)^2]} \end{aligned} \right\} \quad (5)$$

where  $V_0'$  is the strength and  $1/\sqrt{\beta'}$  the range of the 3-body force. Since the contribution of  $V^{3\text{ body}}$  to the binding energy of the nucleus should be of opposite sign to that of  $V^{2\text{ body}}$ ,  $V_0'$  will be chosen to be positive. The

<sup>16</sup> SIEMENS, v.: Diss. Universität München 1958.

total binding energy now becomes

$$E_B^{\text{total}} = E_B^{2\text{body}} + \langle V^{3\text{body}} \rangle. \quad (6)^*$$

$\langle V^{3\text{body}} \rangle$  is a function of the constants  $V'_0, \beta'$  and the radius parameter  $\alpha$ . Conditions I and II now read

$$\left. \frac{E_B^{2\text{body}}}{\partial \alpha} \right|_{\alpha_0} + \frac{\partial}{\partial \alpha} \langle V^{3\text{body}} \rangle \Big|_{\alpha_0} = 0 \quad (7)$$

and

$$E_B^{2\text{body}} + \langle V^{3\text{body}}(\alpha_0) \rangle = E_B(\text{expt}). \quad (8)$$

Writing the Slater determinant representing  $\Psi$  as

$$\Psi = \frac{1}{\sqrt{A!}} \{u_1, u_2 \dots u_A\} \quad (9)$$

where the  $u$ 's are the single particle wave functions of (2), we obtain

$$\begin{aligned} \langle V^{3\text{body}} \rangle &= \int \Psi^* V^{3\text{body}} \Psi d\tau \\ &= \sum_{\substack{i=A-2 \\ i < j < k \\ i=1}}^{i=A-2} \int u_i^*(1) u_j^*(2) u_k^*(3) V(123) \begin{vmatrix} u_i(1) & u_j(1) & u_k(1) \\ u_i(2) & u_j(2) & u_k(2) \\ u_i(3) & u_j(3) & u_k(3) \end{vmatrix} \cdot d\tau. \end{aligned} \quad (10)$$

The right hand side of (10) consists of  $\binom{A}{3}$  summands, where each summand is again a sum of 6 integrals of the three row determinant. On looking at the forms of the  $u$ 's [eq. (2)] and  $V^{3\text{body}}$  [eq. (5)] we find that the matrix elements  $\langle V^{3\text{body}} \rangle$  will be sum of integrals of the general form

$$I = V'_0 \int B e^{-\alpha(r_1^2 + r_2^2 + r_3^2)} e^{-\beta'[(r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_1 - r_3)^2]} \cdot d\tau_1 d\tau_2 d\tau_3 \quad (11)$$

where

$$B = \alpha^n x_1^{n_1} x_2^{n_2} x_3^{n_3} y_1^{n_4} y_2^{n_5} y_3^{n_6} z_1^{n_7} z_2^{n_8} z_3^{n_9} \quad \text{with} \quad n = \frac{1}{2} \sum_{i=1}^9 n_i \quad (12)$$

and

$$\begin{aligned} d\tau_1 &= dx_1 dy_1 dz_1 \\ (r_1 - r_2)^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ r_1^2 &= x_1^2 + y_1^2 + z_1^2 \quad \text{etc.} \end{aligned}$$

The evaluation of these integrals is rather lengthy and tedious but otherwise not difficult. In App. I a table for  $I$  for different values of  $B$  is given. It is found that  $I$  and hence  $\langle V^{3\text{body}} \rangle$  can be expressed as a

\* Different from the usual notation we call "binding energy" the expectation value of the Hamiltonian which is for bound states a negative quantity.

polynomial in  $v = \frac{\alpha}{\alpha + 3\beta'}$ . In Table 1 the values  $\langle V^3 \text{body} \rangle$  thus calculated are given for nuclei  $A = 4, 5, 15, 16, 17$  and also an approximate value for  $A = 40$ . Values for  $A = 11$  and 13 are also included using only one determinant  $\Psi$ , which is, however, not quite justified. The approxima-

Table 1

| Nucleus                         | $V^3 \text{body}$                                                |
|---------------------------------|------------------------------------------------------------------|
| He <sup>4</sup>                 | $4 V'_0 v^3$                                                     |
| He <sup>5</sup> Li <sup>5</sup> | $V'_0 (3v^4 + 7v^3)$                                             |
| B <sup>11</sup> C <sup>11</sup> | $V'_0 (12,84v^6 + 56,31v^5 + 72,40v^4 + 16,64v^3)$               |
| C <sup>12</sup>                 | $V'_0 (21,54v^6 + 75,56v^5 + 89,72v^4 + 22,19v^3)$               |
| C <sup>13</sup> N <sup>13</sup> | $V'_0 (23,86v^6 + 95,58v^5 + 108,12v^4 + 24,58v^3)$              |
| N <sup>15</sup> O <sup>15</sup> | $V'_0 (56,18v^6 + 162,62v^5 + 161,72v^4 + 39,98v^3)$             |
| O <sup>16</sup>                 | $V'_0 (84,48v^6 + 218,72v^5 + 205,36v^4 + 51,55v^3)$             |
| O <sup>17</sup> F <sup>17</sup> | $V'_0 (20,37v^7 + 123,05v^6 + 245,46v^5 + 235,98v^4 + 55,99v^3)$ |
| Ca <sup>40</sup>                | (approx.) $V'_0 (2108v^5 + 1532v^4 + 388v^3)$                    |

N. B. The nuclei He<sup>5</sup> and Li<sup>5</sup> are not stable.

tions involved in the calculation of  $\langle V^3 \text{body} \rangle$  for  $A = 40$  (Ca<sup>40</sup>) are discussed in App. II. Since  $V^3 \text{body}$  is assumed to be charge independent it depends only on the total isotopic spin  $T$  of the nucleus.

**Determination of  $V'_0$  and  $\beta'$ .** As already mentioned our aim is to fix the strength and range of the three-body potential in such a way that conditions (3) and (4) are simultaneously fulfilled. However, this is not possible. The failure is illustrated in Fig. 1 for O<sup>16</sup>. The justification for using O<sup>16</sup> as a test nucleus is that its wave function can be sufficiently accurately represented by a single determinant; configurational mixing should not play any important role for a doubly closed shell nucleus like O<sup>16</sup>, as long as the interaction is of the "weak type". The curve I gives the set of values of  $V'_0$  and  $\beta'$  which fulfill condition (3) (stable  $\alpha = \text{empirical } \alpha_0$ ); curve II gives the relationship between  $V'_0$  and  $\beta'$  required to fulfill condition (4) (correct B. E. for  $\alpha = \alpha_0$ ). The point of intersection of the two curves would have given the desired values of the constants for the 3-body potential, but the curves do *not* intersect for any value of  $V'_0$  or  $\beta'$ . This also becomes evident when we try a numerical solution of the two corresponding equations for O<sup>16</sup>. The situation is similar for nuclei  $A = 15, 17$  etc.

It becomes thus clear that a solution of our original aim viz. prevention of nuclear collapse and correct B. E. at empirical radius, is not possible with the help of the chosen weak type three-body forces. We can, however, remove either of the two difficulties by suitably adjusting  $V'_0$  and  $\beta'$ .

To do this we shall in the following fix  $V'_0$  and  $\beta'$  in such a manner that correct B. E. is obtained for  $\alpha = \alpha_0$  and then try to find the resulting stable radius. In order to find the optimum values of  $V'_0$  and  $\beta'$  we have

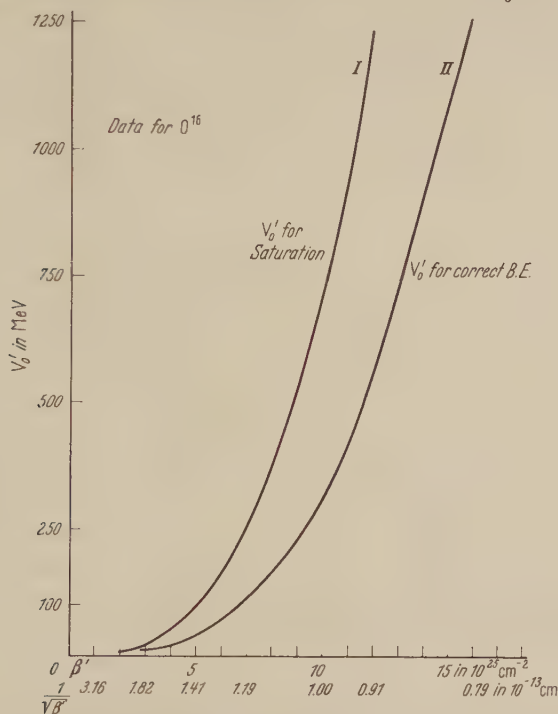


Fig. 1. Graphs showing the strength of the 3-body potential  $V'_0$  required for saturation and correct binding energy respectively, as a function of its range (range  $\propto 1/\sqrt{\beta'}$ ). Curve I gives the set of values of  $V'_0$  and  $\beta'$  which fulfill the condition: Stable radius = empirical radius. Curve II shows the relationship between  $V'_0$  and  $\beta'$  for correct binding energy at empirical radius

considered the nuclei  $O^{15}$ ,  $O^{16}$  and  $F^{17}$  as test nuclei and obtained for  $\beta' \simeq \beta$  ( $1/\sqrt{\beta'}$  = range of two body forces,  $\beta = 4.16 \times 10^{25} \text{ cm}^{-2}$ ) the mean value

$$\left. \begin{aligned} V'_0 &= 138.6 \text{ MeV} \\ \beta' &= 8 \times 10^{25} \text{ cm}^{-2} \end{aligned} \right\} \quad (13)$$

The assumption  $\beta' \simeq \beta$ , in the absence of any first hand principle, appears to be quite plausible from a phenomenological point of view. Moreover such a choice of  $\beta'$  gives  $V'_0$  of the same order of magnitude as  $V_0$ , the depth of two-body potential. This is necessary in order that our 1st order perturbation calculation may have any sense. Thus a certain amount of arbitrariness in the choice of  $V'_0$  and  $\beta'$  could not be avoided; in order to get an idea how the results depend on this choice we shall in the next section consider the effect of varying the strength and range of the 3-body potential.

III. Summary of Results

Having fixed the optimum values of  $V'_0$  and  $\beta'$ , the contribution of  $\langle V^3 \text{ body} \rangle$  to the binding energy and hence  $E_B^{\text{total}}$  ( $E_B^2 \text{ body}$  being known from the calculations of v. SIEMENS) can be easily obtained as a function of  $\alpha$ . In order to exhibit the effect of  $V^3 \text{ body}$  more clearly we have taken

two other values of  $V'_0$ , one higher and one lower than that given above, viz.

$$V'_0 = 217 \text{ MeV}$$

and

$$V'_0 = 94 \text{ MeV}$$

keeping  $\beta'$  fixed at  $8 \times 10^{25} \text{ cm}^{-2}$  and calculated  $E_B^{\text{total}}(\alpha)$  with these three different values of  $V'_0$  for the five nuclei with  $A = 15, 16, 17$ . The results of our calculation are given in Table 2 and Figs. 2 to 4.

In column 1 of Table 2 are listed the five nuclei studied. Column 2 gives the empirical values of  $\alpha(\alpha_0)$  obtained from the Coulomb energy difference of mirror nuclei, the two mirror nuclei are assumed to have

the same radius. For  $\text{O}^{16}$  the value of  $\alpha_0$  is determined from interpolation and is almost the same as for  $A = 15$  and 17. In columns 3 to 5 are given the equilibrium values of  $\alpha$  for the three values of  $V'_0$  obtained from the maxima of the graphs 2 to 4, where we have plotted the variation of  $E_B^{\text{total}}$  with  $\alpha$  for different nuclei; variation of  $E_B^2 \text{ body}$  with  $\alpha$  is also shown for comparison. One notices a marked improvement in all cases

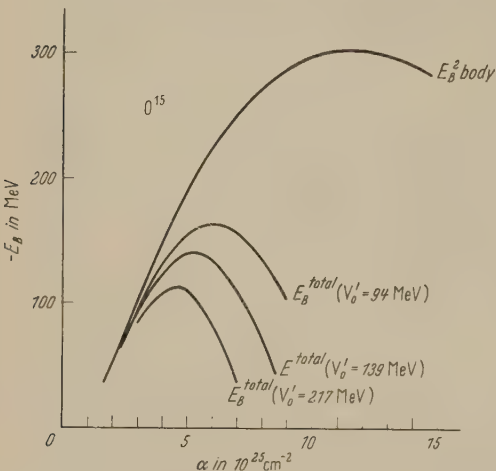
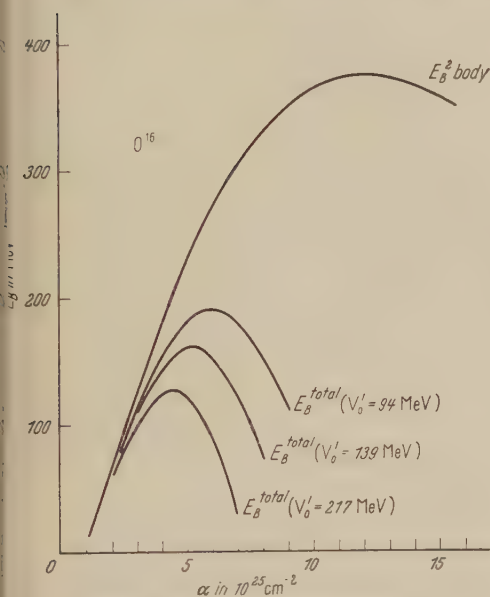
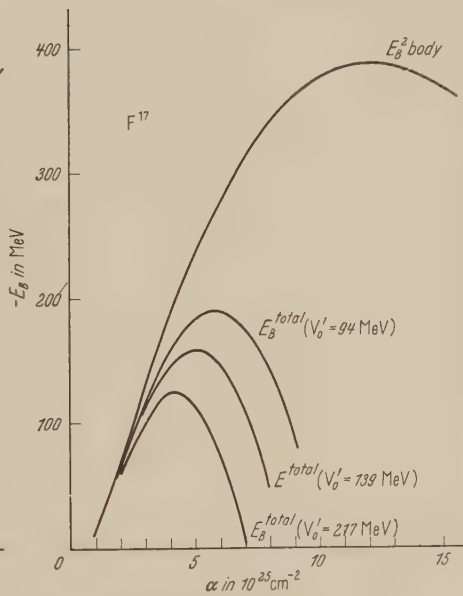


Fig. 2. Variation of calculated binding energy of  $\text{O}^{15}$  with radius parameter  $\alpha$  for 2-body forces alone and for 2-body plus 3-body forces of three different strengths. The nuclear radius is proportional to  $1/\sqrt{\alpha}$ . The values of stable radii are obtained in different cases from the maxima of the curves

Table 2

| 1               | 2                                                  | 3                                                                                                       | 4       | 5       | 6                                                  | 7       | 8       | 9                                                 | 10      | 11      | 12                     |
|-----------------|----------------------------------------------------|---------------------------------------------------------------------------------------------------------|---------|---------|----------------------------------------------------|---------|---------|---------------------------------------------------|---------|---------|------------------------|
| Nucleus         | empirical $\alpha$<br>in $10^{25} \text{ cm}^{-2}$ | Equilibrium<br>$\alpha$ for 2+3-body forces<br>$(\alpha')$ for $V'_0 =$ (in $10^{25} \text{ cm}^{-2}$ ) |         |         | $E_B^{\text{total}}(\alpha_0)$ in MeV for $V'_0 =$ |         |         | $E_B^{\text{total}}(\alpha')$ in MeV for $V'_0 =$ |         |         | $E_B$ (expt)<br>in MeV |
|                 |                                                    |                                                                                                         |         |         |                                                    |         |         |                                                   |         |         |                        |
|                 |                                                    | 94 MeV                                                                                                  | 139 MeV | 217 MeV | 94 MeV                                             | 139 MeV | 217 MeV | 94 MeV                                            | 139 MeV | 217 MeV |                        |
| O <sup>15</sup> | 3,568                                              | 6,0                                                                                                     | 5,5     | 4,5     | -139                                               | -131    | -116    | -162,4                                            | -137,4  | -110,4  | -111,97                |
| N <sup>15</sup> | 3,568                                              | 6,0                                                                                                     | 5,5     | 4,5     | -142,5                                             | -134,5  | -119,5  | -166,9                                            | -141,73 | -114,3  | -115,45                |
| O <sup>16</sup> | 3,57                                               | 6,0                                                                                                     | 5,9     | 4,4     | -155                                               | -131    | -117    | -191,3                                            | -161    | -127    | -127,562               |
| F <sup>17</sup> | 3,565                                              | 5,8                                                                                                     | 5,0     | 4,5     | -138                                               | -131    | -116,5  | -190,5                                            | -158,3  | -128,63 | -128,156               |
| O <sup>17</sup> | 3,565                                              | 5,8                                                                                                     | 5,0     | 4,5     | -141,55                                            | -134,55 | -120,0  | -195                                              | -162,51 | -127,63 | -131,705               |

as a result of the introduction of 3-body forces. The equilibrium radius lies now much closer to the empirical radius. For a suitably high value of  $V'_0$ , the stable  $\alpha$  will equal  $\alpha_0$ , the empirical value, and thus fulfil condition (3); however condition (4) will then be strongly violated. In columns 6 to 8 we give the values of  $E_B^{\text{total}}(\alpha_0)$ , total binding energies for empirical radius. Columns 9 to 11 give the values of  $E_B^{\text{total}}(\alpha')$ , total B. E. for stable  $\alpha$ 's and column 12 the experimental B. E.'s for comparison. Here also we notice marked improvements in all cases.  $E_B^{\text{total}}(\alpha')$

Fig. 3. The same results as in Fig. 2 plotted for  $O^{16}$ Fig. 4. The same results as in Fig. 3 plotted for  $F^{17}$ 

$\left[ \alpha' = \text{stable } \alpha, \text{ for which } \frac{\partial E_B^{\text{total}}}{\partial \alpha} = 0 \right]$  is no longer so widely different from  $E_B(\text{expt})$ . In fact it is possible to find a value of  $V'_0$  for which

$$\left. \begin{aligned} E_B^{\text{total}}(\alpha') &= E_B(\text{expt}) \\ \frac{\partial E_B^{\text{total}}}{\partial \alpha} \Big|_{\alpha'} &= 0. \end{aligned} \right\} \quad (14)$$

We have solved these equations for  $O^{16}$  (see Fig. 5) and obtained after a rather lengthy calculation

$$\alpha' = 4,35 \times 10^{25} \text{ cm}^{-2}$$

$$V'_0 = 217 \text{ MeV}.$$

This  $\alpha'$  corresponds to a nuclear radius  $R \simeq 2,39 \times 10^{-13}$  cm (according to MAYER-JENSEN<sup>17</sup>) while the experimental value of the radius of  $O^{16}$  is  $R(\text{expt}) \simeq 3,02 \times 10^{-13}$  cm.

Instead of trying different "strengths"  $V'_0$ , keeping the range fixed, one might also investigate the effect of  $V^{3\text{body}}$  for different ranges. For values of  $\beta'$  smaller than  $8 \times 10^{25} \text{ cm}^{-2}$ ,  $\alpha'$  (the equilibrium  $\alpha$ ) lies farther off from the empirical  $\alpha$ . For smaller ranges of the 3-body force (i.e. larger values of  $\beta'$ ) the values of  $V'_0$  come out significantly larger than

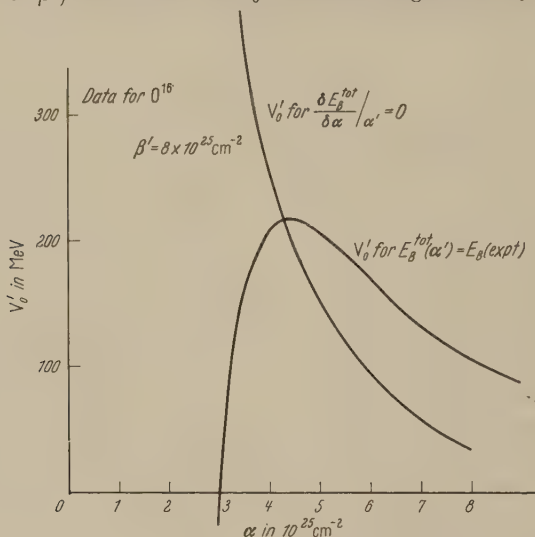


Fig. 5. Curves for  $O^{16}$  for determining the strength of 3-body potential  $V'_0$  which gives correct binding energy at stable radius. The stable radius, however, is not the empirical radius here

the two-body depth. For example for (3-body range)  $= \frac{1}{3}$  (2-body range),  $V'_0 \simeq 1,7 \times 10^5$  MeV and for (3-body range)  $= \frac{1}{2}$  (2-body range),  $V'_0 \simeq 1260$  MeV.

To see how the influence of the three body forces varies with mass number we have made an approximate estimation of  $\langle V^{3\text{body}} \rangle$  for  $Ca^{40}$  and found that  $\langle V^{3\text{body}} \rangle$  lies between 33 and 111 MeV, whereas the discrepancy between the two-body and the experimental values of B. E. is about 187 MeV. Thus the contribution of  $V^{3\text{body}}$  remains quite reasonable with increasing mass number and improves the binding energy compared to the two-body values obtained by v. SIEMENS<sup>10</sup>. Details of this calculation are given in Appendix II.

#### IV. Conclusion and Discussion

From a study of the empirical nuclear data, the necessity of a repulsive force in addition to the usual two-body forces to prevent nuclear collapse is realized. We have considered to what extent the necessary

<sup>17</sup> MAYER, M. G., and J. H. D. JENSEN: Elementary Theory of Nuclear Shell Structure. New York: John Wiley 1955.

repulsion can be supplied by a repulsive three-body potential of suitable range and strength. As test nuclei we have considered the nuclei  $A = 15, 16, 17$  and  $40$ .

Our calculations show that the problem of nuclear saturation can be only *partly* solved with the help of the chosen three-body forces. It is not possible to obtain the correct radius as well as the correct B. E. of a nucleus simultaneously, although either of these may be obtained by a suitable choice of the constants of  $V^{\text{3 body}}$ . This results favours again the temptation that a hard core exists in the nuclear interaction and is mostly responsible for nuclear saturation. It is quite possible that the three-body forces exist in addition to the hard core and that saturation really is a resultant effect of two-body forces including hard core and non-singular three-body forces. We have investigated the effect of a "weak interaction" type three-body force. If this force is chosen to have a strongly singular, repulsive character then much more complicated calculations would be necessary as in the case of singular two-body forces (Brueckner method). However, before introducing such three-body forces of a "strong interaction" type, one should examine in more detail the role of a hard core in the two-body interaction.

Our calculations have shown that the contribution of regular three-body forces lies, even for higher values of the mass number (up to  $A = 40$ ), within the limit of discrepancy between experimental B. E. and that calculated from attractive two-body forces.

Furthermore, the above calculations are based on first order perturbation theory where uncorrelated single particle wave functions are used. The introduction of correlations in the wave function due to the repulsive three-body forces would have several consequences. First, the volume of the nucleus increases, as the nucleons are prevented from coming very close to each other. Secondly, with correlated  $\Psi$ 's higher binding energies (in magnitude) and hence higher values of  $V'_0$  (fig. 1, curve II) would result. On the other hand correlation would have the opposite tendency of increasing the kinetic energy of the nucleons and hence diminishing B. E. It might well happen that the nett effect would be such that curves I and II in fig. 1 would intersect and thereby solve the problem completely. However the effect of configurational mixing (i.e. use of correlated  $\Psi$ 's) is not likely to be very important for closed shell and "closed shell  $\pm 1$ " nuclei like those considered here and therefore it is expected that those corrections would not change the conclusions drawn in this investigation on the effect of three-body forces of the "weak interaction" type.

The author wishes to acknowledge his indebtedness to Dr. F. BECK for suggesting the problem and many helpful discussions. He is thankful to Professor FRITZ BOPP for his kind interest in the progress of the work. His thanks are also due to the Alexander-von-Humboldt Foundation and the Bavarian Government who supported this work with scholarships.

## Appendix I

## Fundamental Integrals

| $B$ |                              | $I = \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \iiint_B \cdot e^{-\alpha(x_1^2 + x_2^2 + x_3^2)} e^{-\beta'[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_1 - x_3)^2]} dx_1 dx_2 dx_3$ |
|-----|------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1.  | 1                            | $\frac{\alpha}{\alpha + 3\beta'} = v$                                                                                                                                           |
| 2.  | $\alpha x_1^2$               | $\frac{1}{6} (2v^2 + v)$                                                                                                                                                        |
| 3.  | $\alpha x_2 x_3$             | $\frac{1}{6} (-v^2 + v)$                                                                                                                                                        |
| 4.  | $\alpha^2 x_1^4$             | $\frac{1}{12} (4v^3 + 4v^2 + v)$                                                                                                                                                |
| 5.  | $\alpha^2 x_3^2 x_2^2$       | $\frac{1}{12} (2v^3 + 0 + v)$                                                                                                                                                   |
| 6.  | $\alpha^2 x_1^2 x_2 x_3$     | $\frac{1}{12} (0 - v^2 + v)$                                                                                                                                                    |
| 7.  | $\alpha^2 x_1^3 x_2$         | $\frac{1}{12} (-2v^3 + v^2 + v)$                                                                                                                                                |
| 8.  | $\alpha^3 x_1^2 x_2^2 x_3^2$ | $\frac{1}{72} (4v^4 + 6v^3 - 6v^2 + 10v)$                                                                                                                                       |
| 9.  | $\alpha^3 x_1^3 x_2^3$       | $\frac{1}{144} (-28v^4 + 12v^3 + 6v^2 + 10v)$                                                                                                                                   |
| 10. | $\alpha^3 x_1^4 x_2^2$       | $\frac{1}{144} (32v^4 + 0 + 12v^2 + 10v)$                                                                                                                                       |
| 11. | $\alpha^3 x_1^4 x_2 x_3$     | $\frac{1}{144} (8v^4 - 24v^3 + 6v^2 + 10v)$                                                                                                                                     |
| 12. | $\alpha^3 x_1^3 x_2^2 x_3$   | $\frac{1}{144} (-4v^4 + 0 - 6v^2 - 10v)$                                                                                                                                        |
| 13. | $\alpha^4 x_1^4 x_2^2 x_3^2$ | $\frac{1}{432} (-22v^5 + 40v^4 - 6v^3 - 20v^2 + 35v)$                                                                                                                           |
| 14. | $\alpha^4 x_1^3 x_2^2 x_3^2$ | $\frac{1}{432} (-16v^5 - 14v^4 - 30v^3 - 35v^2 + 35v)$                                                                                                                          |
| 15. | $\alpha^4 x_1^4 x_2^4$       | $\frac{1}{432} (341v^5 - 32v^4 + 48v^3 + 40v^2 + 35v)$                                                                                                                          |

## Appendix II

Estimation of  $\langle V^3 \text{body} \rangle$  for  $\text{Ca}^{40}$ 

An exact calculation of  $V^3 \text{body}$  for a nucleus containing 40 nucleons, involves the evaluation of  $\binom{40}{3} = 9320$  integrals, each quite lengthy. We have therefore used the following simplifications and approximations for a rough estimation.

$\text{Ca}^{40}$  is a doubly closed 3rd shell nucleus: let us denote by  $B$  the 1st and 2nd shells together, and by  $A$  the 3rd shell. Further, let  $a, b, c, d$  stand for the following

levels:

$$\begin{aligned} a &= 3D \text{ comprising } 20 \text{ nucleons} \\ b &= 3S \text{ comprising } 4 \text{ nucleons} \\ c &= 2P \text{ comprising } 12 \text{ nucleons} \\ d &= 2S \text{ comprising } 4 \text{ nucleons.} \end{aligned}$$

Let  $V_{AAB}$  denote the interaction between two A-nucleons and one B-nucleon and so on. Thus we have:

$$\begin{aligned} \langle V^3 \text{ body (Ca}^{40}) \rangle &= V_{AAA} + V_{AAB} + V_{ABB} + V_{BBB} \\ V_{AAB} &= 2280 V_{aac} + 760 V_{aad} + 72 V_{bbc} + 24 V_{bbd} + 960 V_{abd} + 320 V_{abd} \\ V_{ABB} &= 1320 V_{acc} + 120 V_{add} + 960 V_{acd} + 264 V_{bcc} + 24 V_{bdd} + 192 V_{bcd} \\ V_{BBB} &= V^3 \text{ body (O}^{16}) \end{aligned}$$

For a completed shell we can choose instead of the single particle wave functions (2) any other simpler set of functions which are mutually orthonormal. This is justified because the orthonormal set of  $u$ 's may be looked upon as basic vectors in a subspace of the Hilbert space in which orthogonal transformations which commute with  $H$  leave the result unaltered. This simplification has been used already in the calculation of  $\langle V^3\text{-body} \rangle$  for  $O^{16}$ . Thus for the *completed* 3rd shell comprising 24 nucleons we can use the following type of  $u$ 's:

$$u = \sqrt{2} (x^2 - 1/2\alpha) \delta^+ \cdot \alpha f(r) \quad \text{or} \quad 2\alpha f(r) \cdot xy \cdot \delta^+$$

and calculate  $V_{AAA}$  *rigorously*.  $V_{BBB}$  is also known rigorously. In  $V_{AAB}$  and  $V_{ABB}$  terms like  $V_{aac}$  are naturally different for the different choice of nucleons, but as an approximation we have assumed the interaction to be the same irrespective of which  $a$  and  $c$  nucleons are chosen, which is also not very far from the truth. Since  $v$  is of the order of 0,1 we have neglected terms of higher order than  $v^5$  and obtained

$$\begin{aligned} V_{AAA} &= V'_0 (254,43v^5 + 66,61v^4 + 84,36v^3) \\ 2280 V_{aac} &= V'_0 (373v^5 + 225v^4 + 56,25v^3) \\ 760 V_{aad} &= V'_0 (169v^5 + 112,5v^4 + 56,25v^3) \\ 72 V_{bbc} &= V'_0 (-216v^5 + 33,8v^4 + 9,6v^3) \\ 24 V_{bbd} &= V'_0 (122,5v^5 + 59,2v^4 + 16,7v^3) \\ 960 V_{abc} &= V'_0 (347v^5 + 160v^4 + 30,4v^3) \\ 120 V_{abd} &= V'_0 (180v^5 + 128v^4 + 25,6v^3) \\ \hline V_{AAB} &= V'_0 (974v^5 + 600,1v^4 + 194,8v^3) \\ 1320 V_{acc} &= V'_0 (294v^5 + 196v^4 + 49v^3) \\ 120 V_{add} &= V'_0 (53,4v^5 + 53,4v^4 + 13,35v^3) \\ 960 V_{acd} &= V'_0 (320,4v^5 + 213,6v^4 + 35,6v^3) \\ 264 V_{bcc} &= V'_0 (136,5v^5 + 91v^4 + 9,74v^3) \\ 24 V_{bdd} &= V'_0 (26,75v^5 - 21,33v^4 + 18,66v^3) \\ 192 V_{bcd} &= V'_0 (-171v^5 + 128v^4 + 30,8v^3) \\ \hline V_{ABB} &= V'_0 (660v^5 + 660,67v^4 + 157,15v^3) \\ V_{BBB} &= V'_0 (219v^5 + 205v^4 + 51,6v^3). \end{aligned}$$

Finally we obtain after addition

$$\langle V^3 \text{ body (Ca}^{40}) \rangle = V'_0 [2108v^5 + 1532v^4 + 388v^3].$$

Using the optimum values of  $V_0'$  and  $\beta'$  obtained for  $A \simeq 16$  e.g.

$$\begin{aligned} V_0' &= 140 \text{ MeV} \\ \beta' &= 8 \times 10^{25} \text{ cm}^{-2} \end{aligned}$$

the contribution of three-body forces for  $\text{Ca}^{40}$  comes out for two different values of  $\alpha$  as

$$\begin{aligned} \langle V^{\text{3body}}(\text{Ca}^{40}) \rangle &= +111,5 \text{ MeV} \quad \text{for } \alpha = 3 \times 10^{25} \text{ cm}^{-2} \\ &= +32,9 \text{ MeV} \quad \text{for } \alpha = 2 \times 10^{25} \text{ cm}^{-2}. \end{aligned}$$

The value of  $\alpha$  for  $\text{Ca}^{39}$  obtained from the energy difference of mirror nuclei is  $\alpha = 3,02 \times 10^{25} \text{ cm}^{-2}$ . Using this value of  $\alpha$  for  $\text{Ca}^{40}$  v. SIEMENS obtained the binding energy about 187 MeV higher than the experimental value. However, using an improved method of calculation, taking different (i.e. different  $\alpha$ 's) for different shells viz.

$$\left. \begin{aligned} \alpha_{1S} &= 1,98 \\ \alpha_{2P} &= 2,356 \\ \alpha_{3D} &= 3,208 \\ \alpha_{3S} &= 1,98 \end{aligned} \right\} \times 10^{25} \text{ cm}^{-2}.$$

v. SIEMENS obtains a much better value of the binding energy for  $\text{Ca}^{40}$ , but still higher (in magnitude) by about 68 MeV. Thus the inclusion of  $V^{\text{3body}}$  would in either case improve the value of the binding energy obtained from 2-body forces alone.

## Lebenslauf

Am 9. Juli 1929 wurde ich, Saurabh Sanatani, als Sohn des Herrn Prof. Dr. Rasvihary Das in Kalkutta, Indien geboren. Ich besuchte die "Viswa-Bharati School" zu Santiniketan in Bengalen und im Jahre 1944 bestand ich an dieser Schule mein Abschlußexamen. Im Jahre 1948 erwarb ich den Grad eines Bachelor of Science mit Honours in Physics der Universität Kalkutta und im Jahre 1950 den Grad eines Master of Science in reiner Physik an der gleichen Universität.

Danach arbeitete ich vier Jahre lang als Forschungsstipendiat für theoretische Physik unter Herrn Prof. Dr. K.C. Kar, Presidency College, Kalkutta, über die Diracsche Elektronentheorie. Aus dieser Zeit stammen mehrere meiner Veröffentlichungen in Fachzeitschriften.

Im Oktober 1955 erhielt ich von der Alexander von Humboldt-Stiftung ein Forschungsstipendium zum Studium in Deutschland. Seit dieser Zeit arbeite ich am Institut für theoretische Physik der Universität München unter Anleitung von Herrn Prof. Dr. Fritz Bopp.

